Constant-Attitude Thrust Program Optimization

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The analysis for the optimization of a space vehicle thrust program with prespecified directions of thrust is presented. The criteria for optimizing the values of the prespecified thrust directions are also developed. A numerical comparison in vehicle performance is made between a constant-attitude thrust program and the optimum variable direction program by means of a series of Earth-Mars rendezvous and flyby trajectories, using a power-limited propulsion system. The sensitivity of vehicle performance to departures in thrust direction from the optimum prespecified directions is discussed. It is shown that a constant-attitude thrust program with two or more optimized thrust directions is competitive in vehicle performance with the variable direction program for typical interplanetary missions.

I. Introduction

THIS paper considers a simple application of optimal control theory involving a continuous thrust program for space vehicles in which the direction of thrust is constrained to discrete values. The control variables for this problem are the direction and magnitude of thrust. The criteria for optimizing the control variables with respect to a given performance function subject to the constraints of the problem are developed. The criteria for optimizing the values of the prespecified thrust directions are also developed. Both fixed- and variable-magnitude thrust programs are included. A coasting capability is assumed, and hence a switching function appears in the analysis. In the case of power-limited propulsion systems, the modifications to previous studies[1] of optimal variable thrust direction programs are briefly discussed when a constant-attitude program is employed.

The constant-attitude thrust program has a practical importance for interplanetary vehicles, since it is probably the simplest program that can be executed by a sun-oriented space vehicle. Vehicle performance resulting from the use of such a thrust program is compared with the results from a program in which the thrust is optimally directed along the trajectory for various interplanetary mission types. The trajectories used for these purposes are generated from a heliocentric inverse-square model for Earth-Mars rendezvous and flyby missions for a range of flight times and launch dates.

II. The Constant-Attitude Thrust Program

The optimization theory is now applied to a trajectory problem using a constant-attitude thrust profile. The thrust equations in this discussion are couched in the power-limited propulsion terminology,[1] since the numerical examples to follow are for such a system. The results are readily adapted to propulsion systems without the power-limited constraint. The vehicle is assumed to be a point mass traveling in a vacuum and subjected to a conservative force field. The constraining equations for a power-limited system traveling in a potential field V are

$$\dot{\mathbf{v}} + \Delta V - (\beta/\mu c)\alpha \mathbf{l} = \mathbf{0} \qquad (1)$$

$$\dot{\mathbf{r}} - \mathbf{v} = \mathbf{0} \qquad (2)$$

$$\dot{\mu} + (\beta/c^2)\alpha = 0 \qquad (3)$$

Here, the state variables are position and velocity coordinates $\mathbf{r}$ and $\mathbf{v}$, and the normalized vehicle mass μ, $\mu(t_1) = 1$. The control variables are the unit vector $\mathbf{l}$ parallel to the thrust direction and α, which is a normalized power parameter having the value 1 during propulsion periods and 0 during coasting periods. In this treatment, β is considered to be a constant and equal to twice the kinetic power in the exhaust beam divided by the initial mass of the vehicle. (Compare this with Ref. 2 for an exhaust velocity dependent β due to a variation in efficiency in transferring power from the powerplant to the exhaust beam.) The quantity c is the exhaust velocity, and if it is held fixed, the thrust is constant; for this case $\dot{c} = 0$, and the magnitude of c may be adjusted to yield optimal performance. If c is allowed to vary, it becomes a control variable through which the magnitude of the thrust is varied. Here we considered both the constant thrust case and the variable thrust case with no bounds on c. (Compare this with Ref. 1 for a discussion of bounds on c, where the case of discretely varying c has been heuristically discussed.) The thrust acceleration a during a propulsion phase is given by

$$a = \beta/c\mu \qquad (4)$$

and combining this with Eq. (3), one obtains the rocket equation for power-limited flight

$$\frac{1}{\mu_2} = 1 + \frac{1}{\beta}\int_{t_1}^{t_2} \alpha\, a^2 dt \qquad (5)$$

where $\mu_2 = \mu(t_2)$.

In this discussion, the performance function will be taken as

$$J \triangleq \int_{t_1}^{t_2} \alpha\, a^2 dt = \beta\left(\frac{1}{\mu_2} - 1\right) \qquad (6)$$

This particular performance function is useful in mission feasibility analysis. In Ref. 2, the methods whereby the optimal design of a power-limited vehicle and maximum payload estimates may be constructed from given values of J and total propulsion time are discussed in considerable detail. Our primary purpose here is to investigate the increase in J which results from using a constant-attitude thrust program instead of a variable thrust direction program.

The control variable constraints on α and $\mathbf{l}$ in the constant-attitude program may be written as

$$\alpha = 0, 1 \qquad (7)$$

$$\mathbf{l} = \xi_j \qquad j = 1, \ldots, p \qquad (8)$$

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where ξ_j is a member of the set of prespecified thrust directions. To choose optimally the values of the prespecified thrust directions, we treat ξ_j as a state variable subject to the constraint

$$\dot{\xi}_j = \mathbf{0} \qquad j = 1, \ldots, p \tag{9}$$

and for the case where c is fixed

$$\dot{c} = 0 \tag{10}$$

Upon applying optimization theory[3, 4] to this problem, one obtains the well-known Euler-Lagrange equations

$$\ddot{\boldsymbol{\lambda}} + (\boldsymbol{\lambda}^T\nabla)\nabla V = \mathbf{0} \tag{11}$$

and

$$\dot{\rho} - (\beta/\mu^2 c)\mathbf{1}^T\boldsymbol{\lambda} = 0 \tag{12}$$

where $\boldsymbol{\lambda}$ and ρ are the Lagrange multipliers conjugate to the state variables $\mathbf{v}$ and μ, respectively. Defining L as the quantity

$$L = (\mathbf{1}^T\boldsymbol{\lambda}/\mu) - (\rho/c) \tag{13}$$

which is the switching function, the Hamiltonian becomes

$$\mathcal{H} = (\beta/c)L\alpha - \dot{\boldsymbol{\lambda}}^T\dot{\mathbf{r}} - \boldsymbol{\lambda}^T\nabla V \tag{14}$$

which is a constant since we have an autonomous system. It readily follows that the optimal control law is given by

$$\begin{Bmatrix} L > 0 & \alpha = 1 \\ L < 0 & \alpha = 0 \end{Bmatrix} \tag{15}$$

which determines the optimal periods of propulsion and coasting. For $\mathbf{1}$, one finds that

$$\mathbf{1}^T\boldsymbol{\lambda} = \max[\xi_i{}^T\boldsymbol{\lambda}] \tag{16}$$

That is, the optimal thrust direction at any point along the trajectory is that direction taken from the discrete set ξ_j, which is most nearly parallel to $\boldsymbol{\lambda}$. These results are expected, since in the optimal thrust program with variable thrust direction,[1] the optimal thrust direction is found to be along $\boldsymbol{\lambda}$.

If c is considered as a control variable, it can be shown that the exhaust velocity is determined by the relation

$$c = k/\mu\, \mathbf{1}^T\boldsymbol{\lambda} \tag{17}$$

where k is a constant determined by boundary conditions. It also may be readily shown that the switching function in this case is given by

$$L = \mathbf{1}^T\boldsymbol{\lambda}/2\mu \tag{18}$$

It should be observed that L may be positive and negative for this variable thrust program with constant thrust direction, implying that coast periods may occur. This result, although expected, is contrary to the case holding for the optimal variable thrust and variable direction program where no coast periods occur.

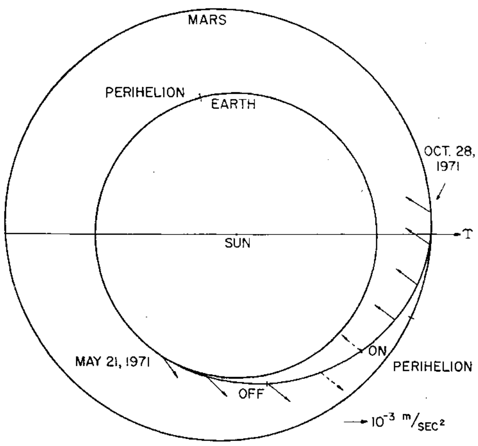

Fig. 1 Variable thrust direction program.

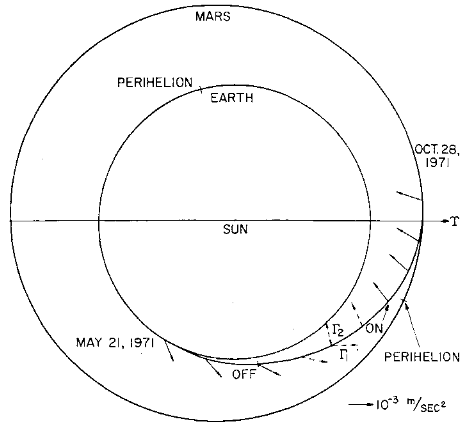

Fig. 2 Constant-attitude thrust program.

If c is considered as a constant, then Eq. (13) holds for L; it may also be generated from the differential equation

$$\dot{L} = \mathbf{1}^T\boldsymbol{\lambda}/\mu \tag{19}$$

which is useful when constant thrust acceleration trajectories are considered, since an infinite c creates some numerical difficulty in using Eq. (13).

It should be pointed out that the foregoing analysis, with the exception of Eqs. (8) and (16), closely parallels some of the analysis contained in Ref. 1, which considered thrust programs with optimal variable direction. The reader is advised to consult Ref. 1 for details that have been omitted here. Moreover, by simply substituting $\mathbf{1}^T\boldsymbol{\lambda}$ at every occurrence of the scalar λ appearing in Ref. 1 and using Eqs. (17) and (25), one nearly has the complete theory for the constant-attitude thrust program.

Finally, for central force fields, there is a constant of integration which is sometimes convenient and given by

$$\mathbf{r} \times \dot{\boldsymbol{\lambda}} - \dot{\mathbf{r}} \times \boldsymbol{\lambda} = \mathbf{K}_1 \tag{20}$$

where $\mathbf{K}_1$ is a constant vector.

A discussion of kinematic boundary conditions and associated transversality conditions will not be given here; a fairly complete discussion is found in Refs. 1 and 5, and these apply also to this constant-attitude problem.

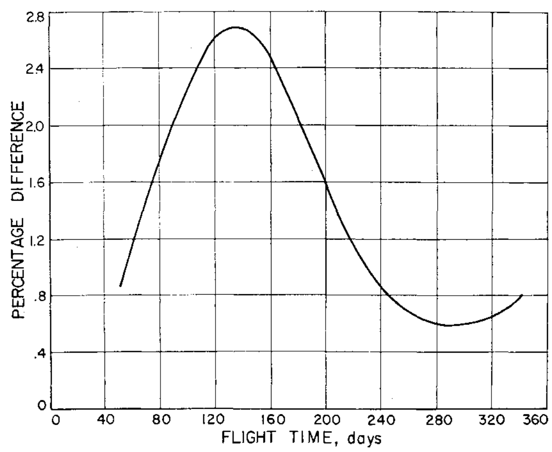

Fig. 3 Percentage increase in *J* using constant-attitude program.

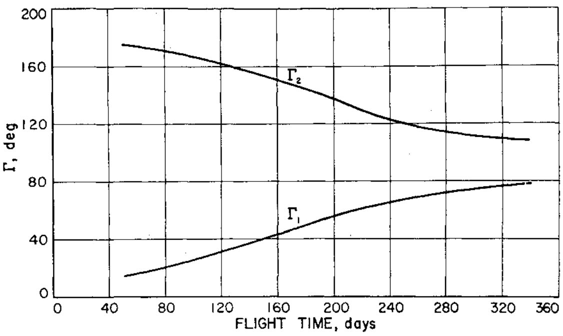

Fig. 4 Optimal thrust direction for $r = 2$.

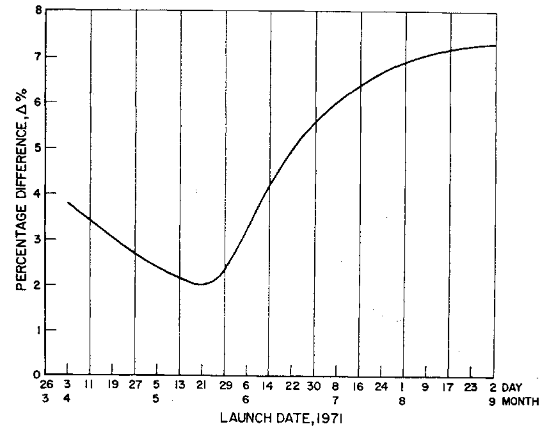

Fig. 6 Percentage increase in J using constant-attitude program.

III. System Parameter Optimization

In the constant-attitude program as formulated, there is the question of the optimal values of the prescribed quantities ξ_j to be employed, and, in the case of constant c, the optimal value of c. The latter problem of determining the optimal value of c to be used in the constant thrust program has been discussed in Ref. 1 for the case of a fixed β and also in Ref. 2 for the general case where β is dependent on c through the efficiency of conversion of power from the powerplant to the exhaust beam. As stated in the previous paragraph, the only modification to the conditions (which were developed in these treatments) for optimal c, which is required for the constant-attitude program, is the replacement of λ by $\mathbf{1}^T\boldsymbol{\lambda}$ in the appropriate functional forms. For constant β, and by using Eq. (10) and the transversality condition, one finds that the condition for optimal c is given by

$$\int_{t_1}^{t_2} \alpha\left(2L - \frac{\mathbf{1}^T\boldsymbol{\lambda}}{\mu}\right) dt = 0 \qquad (21)$$

Similarly, for each ξ_j, it may be shown that the relations

$$\int_{t_1}^{t_2} aa(\boldsymbol{\lambda} \times \mathbf{1})\eta_j dt = \mathbf{0} \qquad j = 1, \ldots, r \qquad (22)$$

are the conditions for optimum ξ_j. The quantity η_j has the value 1 during those phases where $\mathbf{1} = \xi_j$ and the value of zero otherwise. The condition for optimality presented in Eq. (22) is expected; it should be noted that, in the variable direction program where $\mathbf{1}$ varies continuously, the integrand of Eq. (22) is zero at every point along the trajectory.

IV. Comparison of Constant- and Variable-Attitude Thrust Programs

To provide a numerical comparison between these two thrust programs, a series of two-dimensional interplanetary trajectories has been generated. Constant thrust magnitude has been used for both thrust programs, and the coast periods have been optimized by satisfying Eq. (21).

Figure 1 shows an Earth-Mars rendezvous trajectory with a 160-day heliocentric flight time. The planetary configurations correspond to a launch date of May 21, 1971, which is very nearly an optimum launch date for this flight time and synodic era.[5] The thrust direction is optimally varied along this trajectory as determined by the direction of $\boldsymbol{\lambda}$; both the thrust direction and the coast period are shown. The lengths of the thrust vectors are proportional to the thrust acceleration along the trajectory. The value of β was adjusted, so that the exhaust velocity is 50 km/sec.

Figure 2 shows the same mission accomplished with a constant-attitude thrust program in which two thrust directions were allowed. The thrust directions are fixed relative to the heliocentric radius vector, which renders this thrust program highly adaptable to a sun-oriented attitude control system. The choice of thrust direction at any point on the trajectory is determined by Eq. (16), and the two directions themselves have been optimized in the sense that Eq. (22) is satisfied for both thrust directions, resulting in a minimum J.

A series of Earth-Mars rendezvous trajectories for a range of flight times was run using both thrust programs. The kinematic boundary conditions are such that the trajectories commence and terminate at the optimum points on the orbits of the Earth and Mars, so that J is minimized. Figure 3 shows the percentage excess in J over the variable direction program, which results from using a constant-attitude program with two allowed directions relative to the heliocentric radius vector. The variation of J with flight time for the variable direction program is shown in Fig. 7 of Ref. 1. Figure 4 shows the variation of the optimal thrust directions with flight time. The quantity Γ is the angle between the thrust direction and the heliocentric radius vector. For this type of rendezvous trajectory, there is only one subarc

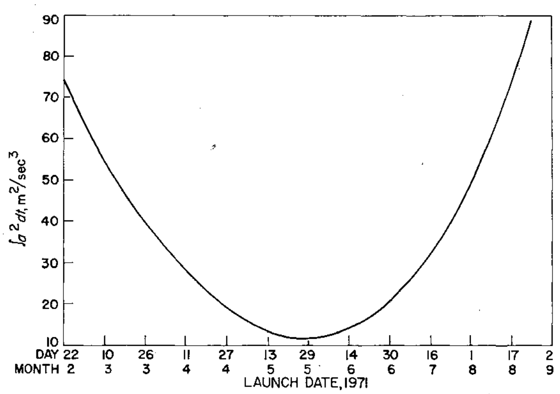

Fig. 5 Variation of J with launch date, variable thrust direction program, 160-day flight time.

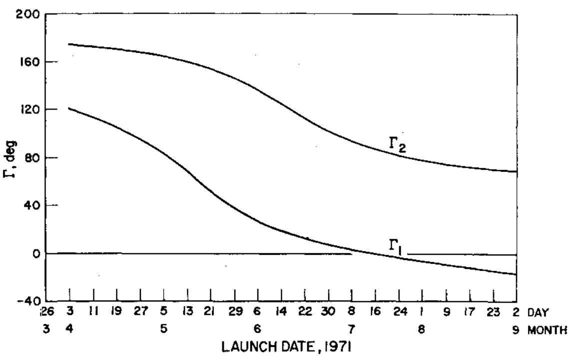

Fig. 7 Optimal thrust direction for $r = 2$.

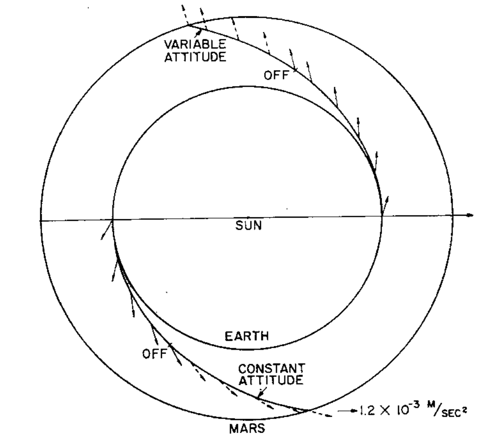

Fig. 8 Optimal Earth-to-Mars flyby trajectories, 123-day flight time, $r(T) = 1.52$ a.u.

for each thrust direction; thus, Γ_1 is used along the initial leg of the trajectory, and Γ_2 holds for the final part.

Another series of Earth-Mars rendezvous trajectories was generated for a range of launch dates holding the flight time fixed. In a similar way as the forementioned, Fig. 5 shows the variation of J with launch date for a fixed 160-day flight time using the variable direction program. Figure 6 shows the excess in J which results from the constant-attitude program, and Fig. 7 gives the variation of Γ_1 and Γ_2 with launch date for this program. The percentage difference in J is, in general, larger in this last example. This is because of the greater maneuvering in optimal thrust direction which occurs when launch dates that are radically apart from the optimal launch date are used (cf. Fig. 2 of Ref. 5).

As a third example, a set of Earth-Mars flyby trajectories is presented using the two thrust programs. In this case, only one direction relative to the radius vector is allowed for the constant-attitude program, but the direction is optimized. Figure 8 shows the trajectories for a 123-day mission as obtained from these two thrust programs. The angular position θ and the two components of velocity at encounter are unspecified, and the resulting transversality conditions[1] yielding the optimal angular position and final velocities are satisfied. Figure 9 gives the excess in J over the variable-attitude value and the optimal Γ_1 resulting from using the constant-attitude program. The variation of J with flight time for the variable-attitude program is found in Fig. 28 of Ref. 6.

It is interesting to inquire about the sensitivity of J to departures of the ξ_j from their optimal values. It may be shown, for the two-dimensional constant-attitude program, that the first variation in J with respect to Γ_j becomes

$$\frac{\partial J}{\partial \Gamma_j} = \frac{\beta}{\mu_2{}^2\rho(t_2)} \int_{t_1}^{t_2} a|\boldsymbol{\lambda} \times \mathbf{1}|\eta_j dt \tag{23}$$

The analytic form for the second variation is considerably more complex, but the dominant term, evaluated where

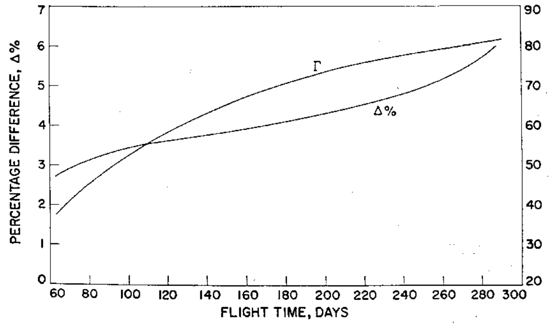

Fig. 9 Percentage increase in J using constant-attitude program, optimal thrust direction for $r = 1$.

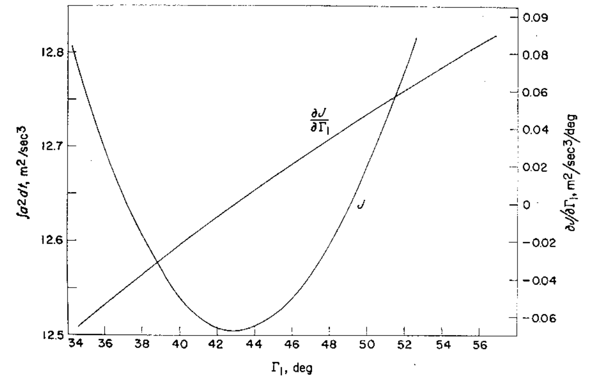

Fig. 10 Sensitivity of J to suboptimal thrust direction.

Eq. (23) is zero, is given by

$$\partial^2 J/\partial \Gamma_j{}^2 \sim 2J \tag{24}$$

where Γ is evaluated in radians. This last equation suggests that the variation in J with Γ is rather flat, since Γ is expressed radian measure.

As an example, Fig. 1 exhibits the variation of J with Γ_1 for the 160-day mission as presented in Fig. 2. Also shown is Eq. (23), which, of course, crosses through zero as the minimum value of J. Since J possesses a stationary value at the optimum Γ_1, it varies in a second-order manner with Γ_1, and Fig. 10 verifies that the sensitivity of J to the choice of Γ_1 is not particularly critical. Similar considerations also apply to Γ_2.

V. Summary and Conclusions

It has been shown that the constant-attitude thrust program with optimized thrust directions is competitive in vehicle performance with the variable direction program. By the use of two optimized thrust directions, the increase in J for rendezvous trajectories departing near the optimum launch date is only 1 or 2%. Furthermore, the use of three or more allowed thrust directions gains very little in performance; the use of only one thrust direction for rendezvous trajectories is generally extremely inefficient, and in many cases the mission cannot be accomplished. For flyby missions, the use of only one thrust direction only slightly degrades the vehicle performance; this is because the variation of the optimal direction of thrust for typical flyby missions is much less radical than in rendezvous missions. From the computational standpoint, the variable direction thrust program is more convenient, since in resolving the two-point boundary problem it is not necessary in this program to satisfy the conditions of Eq. (22).

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